GUIDED VLASOV BEAMS

Risto Koivula

Department of Mechanical Engineering, Lappeenranta University of Technology, P.O.B
FIN-53851 Lappeenranta

ABSTRACT

The deformations of a non-guided Vlasov beam with a rigid-in-plane open cross-section can be expressed comprehensively by means of four displacement functions: the axial extension $u(x)$, the deflections in the principle directions $v(x)$ and $w(x)$ of the torsional axis which for a non-guided beam coincides with the shear axis, and the angle of twist $\theta_u(x)$ around the torsional axis. These displacements with their associate deformations and force quantities embody, in terms of the generalized beam theory /4/, the **fundamental modes** of a Vlasov beam. The fundamental modes are mutually orthogonal in the sense that axial stresses due to force quantities associated to one mode do not do virtual work in deformations associated to any other mode.

For a **guided** beam the deformations are restricted by longitudinally continuous restraints preventing one or several lateral or axial degrees of freedom. Then the active, the **effective degrees of freedom** are often mutually orthogoal in some other, transformed**,** non-fundamental system of axes and special points, here called the **solving system** of axes and special points. Due to the orthogonality, the displacement functions in the solving system can be applied to an element without giving rise to displacements or distributed support forces associated to any other mode. This is a necessary prerequisite for constructing a stiffness matrix in a system of axes and special points. So the stiffness (sub)matrix for effective degrees of freedom is now formed. It is a (sub)matrix of the well-known stiffness matrix of a Vlasov element using transformed cross-section integrals A, I_v, I_w, I_ω. (Special points have a physical meaning, generic points are arbitrary.)

KEYWORDS

Vlasov beam, Vlasov element, guided beam, sectorial area, system of axes and special points, system of axes and generic points, torsional axis, extensional axis

INTRODUCTION

The theory of guided beams was introduced by V.Z.Vlasov as a part of the theory of thin-walled beams in the form of cases with one lateral degree of freedom, the torsion around a forced torsional axis (cases 1.3 and 2.2 in this article) /4/.

The case with two lateral degrees of freedom (1.1) was introduced for instance in lessons of Professor Herman Parland in Tampere University of Technology about in the year 1975. The case was an angle cross-section, underlying warping torsion as a guided beam.

More detailed introduction of the case of two lateral degrees of freedom was made by Professor Karl Roik in 1978 in / /.

In article /1/ from 1986 is introduced by the author a theory in matrix form, expressing the different cases, and some new, as special cases of one coherent theory. The aim of the present article is to update and complete this theory.

The theory of guided beam can be derived alternatively by means of the generalized beam theory introduced by Richard Schardt /6/. Indeed, in this article the direct method of /1/ to calculate the special points is completed for various cases.

This research was done in the Department of Mechanical Engineering at Lappeenranta University of Technology under Professor of Steel Constructions Erkki Niemi.

DEFINITION OF BASIC NOTIONS

The study is based on deformations of a prismatic thin-walled beam with an open rigid-in-plane cross-section. The shear stresses are supposed to be sufficiently parallel with the profile line for the beam to be regarded as strengthened with transverse rigid-in-plane diaphragms in short intervals. Outside the profile lines an axis denotes the locus of the intersectional points of a straight axial line with the diaphragms in the case of a non-deformed beam, and the deflection of the axis denotes the locus taken by these points in the case of a deformed beam.

The shear deformation at the profile line is negligible:

$$\gamma_{xs}(x,s) = \frac{\partial u(x,s)}{\partial s} + \frac{\partial v_s(x,s)}{\partial x} = 0 \qquad (1)$$

Without restricting the generality, any lateral motion of a cross-section can be expressed by means of two perpendicular deflections $v(x,a_{yi},a_{zi})$ and $w(x,a_{yi},a_{zi})$ of an arbitrary pole axis $A_i = (a_{yi}, a_{zi})$ and an angle of twist $\theta_{ui}(x)$ around this axis. The displacements of an arbitrary point $P(s) = (y(s),z(s))$ of the profile line are now obtained in systems of axes and special points of figure 1 in the following way:

$$v(x,y(s),z(s)) = v(x,a_{yi},a_{zi}) - \theta_{ui}(x)[z(s) - a_{zi}] \tag{2}$$

$$w(x,y(s),z(s)) = w(x,a_{yi},a_{zi}) - \theta_{ui}(x)[y(s) - a_{yi}] \tag{3}$$

The tangential displacement $\upsilon_s(s,x)$ of this point in the notations used in figure 1 is now

$$\upsilon_s(x,s) = -\, v(x,y(s),z(s))\cos\alpha(s) + w(x,y(s),z(s))\sin\alpha(s)$$

$$= -\, v(x,a_{yi}, a_{zi})\cos\alpha(s) + w(x,a_{yi},a_{zi})\sin\alpha(s) - \theta_{ui}(x)h(s), \text{ in which} \tag{4}$$

$$h_i(s) = \bar{r}_i \cdot \bar{i}_t = -\bar{r}_i \cdot \bar{i}_s \times \bar{i}_x = -[z(s) - a_{zi}]\cos\alpha(s) + [y(s) - a_{yi}]\sin\alpha(s) \tag{5}$$

Based on assumption (2) the axial dispacement distribution is obtained by

$$u(x,s) = \int[v'(x,a_{yi},a_{zi})\cos\alpha(s) - w'(x,a_{yi},a_{zi})\sin\alpha(s) + \theta_{ui}'(x)h_i(s)]\mathrm{d}s \tag{6}$$

$$= \int[v'(x,a_{yi},a_{zi})\mathrm{d}y(s) - w'(x,a_{yi},a_{zi})\mathrm{d}z(s) + \theta_{ui}'(x)\mathrm{d}\omega_i(s)] \tag{7}$$

$$= u(x) + v'(x,a_{yi},a_{zi})[y(s) - y_{0i}] - w'(x,a_{yi},a_{zi})[z(s) - z_{0i}] + \theta_{ui}'(x)[\omega_i(s) - \omega_{0i}] \tag{8}$$

$$= u(x) + \theta_{wi}(x,a_{yi},a_{zi})[y(s) - y_{0i}] + \theta_{vi}(x,a_{yi},a_{zi})[z(s) - z_{0i}] + \theta_{ui}'(x)[\omega_i(s) - \omega_{0i}] \tag{9}$$

$$\omega_i(s) = \int_{s_0}^{s} h_i(s)\mathrm{d}s = -\bar{i}_x \cdot \int_{s_0}^{s} \bar{r}_i \times \bar{i}_s \mathrm{d}s = -\bar{i}_x \cdot \int_{s_0}^{s} \bar{r}_i \times \mathrm{d}\bar{s} = -\bar{i}_x \cdot \int_{s_0}^{s} [\bar{p}(s) - \bar{a}_i] \times \mathrm{d}\bar{s} \tag{10}$$

The parameters y_{0i} and z_{0i} now define in an "absolute system of axes" a new generic point called here the origin, because it serves this function in expressions of unit deformations ("warpings") $y - y_{0i}$ and $z - z_{0i}$. A prime (′) denotes derivation with respect to x.

The difference between sectorial areas, calculated around two different poles $A_1 = (a_{y1}, a_{z1})$ and $A_2 = (a_{y2}, a_{z2})$, starting from one sectorial origin, s_0, is obtained from

$$\omega_2(s) - \omega_1(s) = -\bar{i}_x \cdot \int_{s_0}^{s} (-\bar{a}_2 + \bar{a}_1) \times \mathrm{d}\bar{s} = \bar{i} \cdot \Big|_{s_0}^{s} (\bar{a}_2 - \bar{a}_1) \times \bar{p}(s)$$

$$= -\,(a_{y2} - a_{y1})[z(s) - z(s_0)] + (a_{z2} - a_{z1})[y(s) - y(s_0)] \tag{11}$$

Without restricting the generality, the sectorial origins s_0 can be selected so that for each $\omega_i(s)$ is valid

$$\int_A \omega_i(s)dA = 0$$

(13)

Both $\omega_i(s)$ can be now associated with a separate arbitrary ω_{0i}.

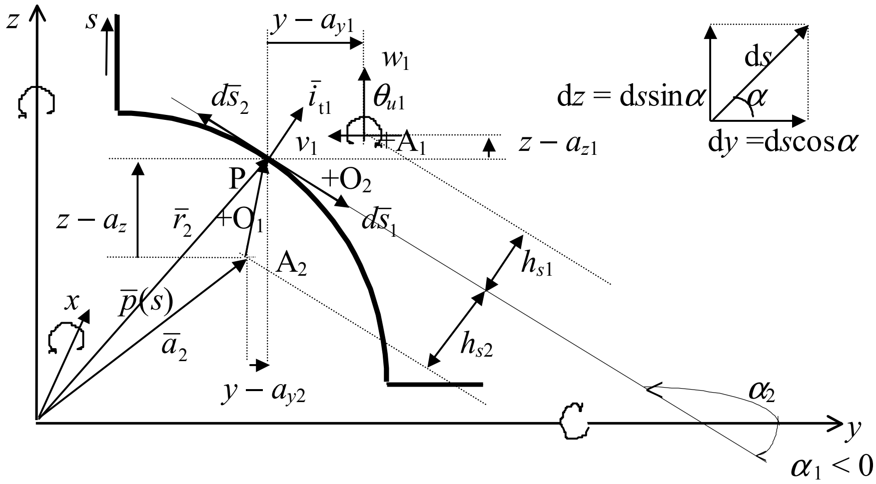

Figure 1. The introduced notations in two separate systems of axes and generic points O_1A_1xyz and O_2A_2xyz. The displacements $u_i(x)$, $v_i(x)$ and $w_i(x)$, the rotations $\theta_{vi}(x) = -w_i'(x)$ and $\theta_{wi}(x) = v_i'(x)$, the angle of twist $\theta_{ui}(x)$, the rate of twist $\theta_{ui}'(x)$ and the force quantities F_{ui}, F_{vi}, F_{wi}, M_{ui}, M_{vi}, M_{wi} and B_i are expressed in a right-handed system of axes *uvw*, but the geometric quantities of the cross-section in a left-handed system *xyz*. Then the system is in accordance with the FEM-program AGIFAP /6/, developed in Lappeenranta University of Technology. The aim of this exceptional choice of systems of axes is to eliminate unnecessary negative signs from the equations between force and displacement quantities.

The axial strains are obtained by differentiating the axial displacements from equation (8) with respect to x

$$\varepsilon(x,s) = u'(x,s) \tag{14}$$

The axial normal stresses are obtained according to Hooke´s law, E is Young´s modulus:

$$\sigma(x,s) = E\varepsilon(x,s) \tag{15}$$

The force quantities in an arbitrary system of axes and generic points $A_iO_ixyzs_{0i}$ are obtained by multiplying the stresses from (19) with the unit deformation functions $1, y - y_{0i}$, $z - z_{0i}$ and $\omega_i(s) - \omega_{0i}$ of the system, and integrating over the cross-section area:

$$F_{ui} = \iint_A \sigma_x \mathrm{d}A = E\iint_A \varepsilon_x \mathrm{d}A = E[\, u_i'\iint_A \mathrm{d}A + \theta_v'\iint_A (z - z_{oi})\mathrm{d}A + \theta_w'\iint_A (y - y_{oi})\mathrm{d}A$$
$$+ \theta_i''\iint_A (\omega_i - \omega_{oi})\mathrm{d}A = E(u_i'A + \theta_{vi}'S_{wi} + \theta_{wi}'S_{vi} + \theta_{ui}''S_{\omega i}) \tag{16.1}$$

$$M_{vi} = E[u_i'\iint_A (z-z_{01})\mathrm{d}A - \iint_A (z-z_{01})^2\mathrm{d}A + \iint_A (z-z_{01})(y-y_{01})\mathrm{d}A +$$

$$\theta_i''\iint_A (\omega-\omega_{0i})(z-z_0)\mathrm{d}A] = E(u_i'S_{vi} + \theta_{vi}'I_{vi} - \theta_{wi}'I_{vwi} + \theta_{ui}''I_{\omega vi}) \qquad (16.2)$$

$$M_{wi} = E\,[u_i'\iint_A (y-y_{0i})\mathrm{d}A - w_i''\iint_A (z-z_{0i})(y-y_{0i})\mathrm{d}A + v_i''\iint_A (y-y_{01})^2\mathrm{d}A +$$

$$\theta_i''\iint_A (\omega_i-\omega_{0i})(y-y_{0i})dA] = E(u_i'S_{wi} - \theta_{vi}'I_{vwi} + \theta_{wi}'I_{wi} + \theta_{ui}''I_{\omega wi}) \quad (16.3)$$

$$B_i = E\,[u_i'\iint_A (\omega_i-\omega_{0i})\mathrm{d}A - w_i''\iint_A (z-z_{0i})(\omega_i-\omega_{0i})\mathrm{d}A + v_i''\iint_A (\omega_i-\omega_{0i})(y-y_{01})\mathrm{d}A +$$

$$\theta_{ui}''\iint_A (\omega-\omega_{0i})^2\mathrm{d}A] = E(u_i'S_{\omega i} + \theta_{vi}'I_{\omega wi} + \theta_{wi}'I_{\omega vi} + \theta_{ui}''I_{\omega i}) \qquad (16.4)$$

The relation between force and displacement quantities in matrix form:

$$\{F\}_i = \begin{Bmatrix} F_u \\ M_v \\ M_w \\ B \end{Bmatrix}_i = E\begin{bmatrix} A & S_v & S_w & S_\omega \\ S_v & I_v & -I_{vw} & I_{v\omega} \\ S_w & -I_{vw} & I_w & I_{w\omega} \\ S_\omega & I_{v\omega} & I_{w\omega} & I_\omega \end{bmatrix}_i \begin{Bmatrix} u' \\ \theta_v{}' \\ \theta_w{}' \\ \theta_u{}'' \end{Bmatrix}_i = E[D]_i\{u\}_i \qquad (17)$$

The cross-section integrals, the elements of the matrix of eq. (17), are given in a right-handed system, as the indices shows. By placing the origin at the centroid O = (y_0,z_0), the pole at at the shear centre A = (a_{y0},a_{z0}), and assuming that $\omega_{0i} = 0$, is obtained

$$\{F\} = \begin{Bmatrix} F_u \\ M_v \\ M_w \\ B \end{Bmatrix} = E\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & I_{v0} & -I_{vw0} & 0 \\ 0 & -I_{vw0} & I_{w0} & 0 \\ 0 & 0 & 0 & I_{\omega 0} \end{bmatrix} \begin{Bmatrix} u' \\ \theta_v{}' \\ \theta_w{}' \\ \theta_u{}'' \end{Bmatrix} = E[D_0]\{u\}$$

$$= E[\Phi(\phi)][\tilde{D}][\Phi(\phi)]^{\mathrm{T}}\{u\} \qquad (18)$$

$$= E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & \tilde{I}_{v0} & 0 & 0 \\ 0 & 0 & \tilde{I}_{w0} & 0 \\ 0 & 0 & 0 & \tilde{I}_{\omega 0} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u' \\ \theta_v{}' \\ \theta_w{}' \\ \theta_u{}'' \end{Bmatrix} \qquad (19)$$

$$\phi = -\frac{1}{2}\arctan\frac{2I_{vw0}}{I_{w0}-I_{v0}} \qquad (20)$$

A circumflex (~) over a symbol, mostly a matrix or vector, denotes quantities given in the fundamental system, a subscript zero ($_0$) denotes quantities in a **centric** (but non-principle) system with the origin at the centroid and pole at the shear centre. The force and displacement symbols in the fundamental system are written without subscripts.

TRANSFORMATION MATRICES OF DISPLACEMENT QUANTITIES

In this chapter the directions of the ”absolute axes” are those of the (centric) principle axes. A number subscript, for instance ($_1$), denotes a quantity given in a non-fundamental, transformed system.

Due to the orthogonality $u(x)$, $v(x)$, $w(x)$ and $\theta_u(x)$ in the fundamental system AOxyz the following distribution of axial displacements can be applied at an end of an element:

$$u(x,y,z) = u(x) + v'(x)(y - y_0) - w'(x)(z - z_0) + \theta_u'(x)\omega(s) \quad (21)$$

$$= u(x) + \theta_w(x)(y - y_0) + \theta_v(x)(z - z_0) + \theta'_u(x)\omega(s) \quad (22)$$

According to equations (3) and (4) the deflections of an arbitrary axis $A_1 = (a_{y1}, a_{z1})$ are

$$v_1(x) = v(x) - (a_{z1} - a_{z0})\theta_u(x) \quad (23)$$

$$w_1(x) = w(x) - (a_{y1} - a_{y0})\theta_u(x) \quad (24)$$

$$v_1'(x) = v'(x) - (a_{z1} - a_{z0})\theta_u'(x) = \theta_w(x) - (a_{z1} - a_{z0})\theta_u'(x) = \theta_{w1}(x) \quad (25)$$

$$w_1'(x) = w'(x) - (a_{y1} - a_{y0})\theta_u'(x) = -\theta_v(x) - (a_{y1} - a_{y0})\theta_u'(x) = -\theta_{v1}(x) \quad (26)$$

Because equation (13) is valid both for fundamental $\omega(s)$ and transformed $\omega_1(s)$ and (y_0,z_0) is the centroid

$$\omega_1(s) - \omega(s) = -(a_{y1} - a_{y0})(z - z_0) + (a_{z1} - a_{z0})(y - y_0) \quad (27)$$

$$= (a_{y1} - a_{y0})(z - z_{01}) - (a_{z1} - a_{z0})(y - y_{01}) + (a_{y1} - a_{y0})(z_{01} - z_0) - (a_{z1} - a_{z0})(y_{01} - y_0) \quad (28)$$

We now place the origin at an arbitrary point (y_{01},z_{01}) and add a constant $-\omega_{01}$ to the sectorial area. The physical meaning of the extension $u_1(x)$ is now turned from a constant distribution with respect to s of axial displacement to a same kind of quantity, not explained by the other terms of the axial displacement distribution (8). Based on equations (25), (26) and (28) the axial displacement (22) is

$$u(x,y,z) = [u(x) + \theta_w(x)(y_{01} - y_0) + \theta_v(x)(z_{01} - z_0) - \theta_u'(x)(a_{z1} - a_{z0})(y_{01} - y_0)$$

$$+ \theta_u'(x)(a_{y1} - a_{y0})(z_{01} - z_0) + \theta_u'(x)\omega_{01}] + [\theta_v(x) - \theta_u'(x)(a_{z1} - a_{z0})](y - y_{01})$$

$$+ [\theta_w(x) + \theta_u'(x)(a_{y1} - a_{y0})](z - z_{01}) + \theta_u'(x)(\omega_1 - \omega_{01}) \quad (29)$$

$$= u_1(x) + \theta_{v1}(x)(z - z_{01}) + \theta_{w1}(x)(y - y_{01}) + \theta_u'(x)(\omega_1 - \omega_{01}) \quad (30)$$

Here one and the same axial displacement distribution is expressed in separate systems of axes and generic points. Based on (29) and (30) this relation is in matrix form:

$$\begin{Bmatrix} u_1 \\ \theta_{v1} \\ \theta_{w1} \\ \theta'_u \end{Bmatrix} = \begin{bmatrix} 1 & z_{01}-z_0 & y_{01}-y_0 & (a_{y1}-a_{y0})(z_{01}-z_0)-(a_{z1}-a_{z0})(y_{01}-y_0)+\omega_{01} \\ 0 & 1 & 0 & a_{y1}-a_{y0} \\ 0 & 0 & 1 & -(a_{z1}-a_{z0}) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ \theta_v \\ \theta_w \\ \theta'_u \end{Bmatrix} \quad (31)$$

or $\{u\}_1 = [T]_{1,0}{}^{\mathrm{T}}\{\tilde{u}\}$ (32)

By inverting the matrix of equation (33) is obtained matrix $[T]_{0,1}{}^{\mathrm{T}}$, which transforms from a non-fundamental system with axes in principle directions to the fundamental system:

$$\{\tilde{u}\} = \begin{Bmatrix} u \\ \theta_v \\ \theta_w \\ \theta'_u \end{Bmatrix} = \begin{bmatrix} 1 & -(z_{01}-z_0) & -(y_{01}-y_0) & -\omega_{01} \\ 0 & 1 & 0 & -(a_{y1}-a_{y0}) \\ 0 & 0 & 1 & a_{z1}-a_{z0} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_{v1} \\ \theta_{w1} \\ \theta'_{u1} \end{Bmatrix} = [T]_{0,1}{}^{\mathrm{T}}\{u\}_1 \quad (35)$$

TRANSFORMATION MATRICES OF FORCE QUANTITIES

If we examine the force quantities defined by the integrals (16.1–4) in the fundamental system A$Oxyz$ and in a transformed system $A_1O_1xyzs_{01}$ in the principal direction:

$$F_{u1} = \iint_A \sigma_x dA = \tilde{F}_u$$

(39.1)

$$M_{v1} = \iint_A \sigma_x (z-z_{01})dA = \iint_A \sigma_x (z-z_0)dA - (z_{01}-z_0)\iint_A \sigma_x dA = \tilde{M}_v - (z_{01}-z_0)\tilde{F}_u \quad (39.2)$$

$$M_{w1} = \iint_A \sigma_x (y-y_{01})dA = \iint_A \sigma_x (y-y_0)dA - (y_{01}-y_0)\iint_A \sigma_x dA = \tilde{M}_w - (y_{01}-y_0)\tilde{F}_u \quad (39.3)$$

$$B_1 = \iint_A \sigma_x (\omega_1 - \omega_{01})dA = \iint_A \sigma_x \left[\omega - (a_{y1}-a_{y0})(z-z_0) + (a_{z1}-a_{z0})(y-y_0) - \omega_{01}\right]dA$$

$$= \iint_A \sigma_x \omega dA - (a_{y1}-a_{y0})\iint_A \sigma_x (z-z_0)dA + (a_{z1}-a_{z0})\iint_A \sigma_x (y-y_0)dA - \omega_{01}\iint_A \sigma_x dA$$

$= \tilde{B} - (a_{y1}-a_{y0})\,\tilde{M}_v + (a_{z1}-a_{z0})\,\tilde{M}_w - \omega_{01}\,\tilde{F}_u$, in matrix form (39.4)

$$\{F\}_1 = \begin{Bmatrix} F_{u1} \\ M_{v1} \\ M_{W1} \\ B_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -(z_{01}-z_0) & 1 & 0 & 0 \\ -(y_{01}-y_0) & 0 & 1 & 0 \\ -\omega_{01} & -(a_{y1}-a_{y0}) & a_{z1}-a_{z0} & 1 \end{bmatrix} \begin{Bmatrix} F_u \\ M_v \\ M_w \\ B \end{Bmatrix} = [T]_{0,1}\{\tilde{F}\} \quad (40)$$

The transformation matrix $[T]_{0,1}$ of eq. (40) is a transpose to the matrix of eq. (35). A transformation matrix from a transformed system in the principle directions to the fundamental system, a transpose of the matrix of (31), is obtained by inverting this matrix.

Generally $$[T]_{i,k}[T]_{j,i} = [T]_{j,k}$$
(42)

By expressing $\{F\}$ and $\{u\}$ in a transformed system $A_1O_1s_{01}xyz$ is obtained

$$\{F\} = [T]_{0,1}^{-\mathrm{T}}\{F\}_1 = E[D_0][T]_{0,1}^{\mathrm{T}}\{u\}_1, \text{ based on which} \quad (43)$$

$$\{F\}_1 = E[T]_{0,1}[D_0][T]_{0,1}^{\mathrm{T}}\{u\}_1 = E[D_1]\{u_1\}$$
(44)

By rotating $\{F\}_1$ and $\{u\}_1$ by angle ϕ into a non-principle system is obtained

$$\{F\}_2 = E\left([\Phi(\phi)]_{2,0}[T]_{0,1}[\Phi(\phi)]_{2,0}^{\mathrm{T}}\right)\left([\Phi(\phi)]_{2,0}[\widetilde{D}_0][\Phi(\phi)]_{2,0}^{\mathrm{T}}\right)\left([\Phi(\phi)]_{2,0}[T]_{0,1}^{\mathrm{T}}[\Phi(\phi)]_{2,0}^{\mathrm{T}}\right)\{u\}_2$$

$$= E[T]_{0,2}[D_0]_2[T]_{0,2}^{\mathrm{T}}\{u\}_2$$
(45)

The bracketed expressions denote tensor rotations of angle ϕ of the matrices $[T]_{i,j}$ and $[D]_i$. So the transformation matrices $[T]$ are formed in the same way independent on the angle ϕ.

The **generalized Steiner´s rule** gives the elements of matrix $[D]_i$ in an eccentric system as functions of the elements of the centric, but usually non-principle matrix $[D_0]$**:**

$$\begin{bmatrix} A & S_v & S_w & S_\omega \\ S_v & I_v & -I_{vw} & I_{v\omega} \\ S_w & -I_{vw} & I_w & I_{w\omega} \\ S_\omega & I_{v\omega} & I_{w\omega} & I_\omega \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -(z_{01}-z_0) & 1 & 0 & 0 \\ -(y_{01}-y_0) & 0 & 1 & 0 \\ -\omega_{01} & -(a_{y1}-a_{y0}) & a_{z1}-a_{z0} & 1 \end{bmatrix} \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & I_{v0} & -I_{vw0} & 0 \\ 0 & -I_{vw0} & I_{w0} & 0 \\ 0 & 0 & 0 & I_{\omega 0} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & -(z_1-z_0) & -(y_1-y_0) & -\omega_{01} \\ 0 & 1 & 0 & -(a_{y1}-a_{y0}) \\ 0 & 0 & 1 & a_{z1}-a_{z0} \\ 0 & 0 & 0 & 1 \end{bmatrix} = [D_1] \quad (46)$$

$$A_1 = A = \iint_A dA$$
(47)

$$S_{v1} = \bar{i}_v \cdot \iint_A \bar{p}_1 \times \bar{i}_u \mathrm{d}A = \iint_A (z - z_{01})\mathrm{d}A = -(z_{01} - z_0)A \quad (48)$$

$$S_{w1} = \bar{i}_w \cdot \iint_A \bar{p}_1 \times \bar{i}_u \mathrm{d}A = \iint_A (y - y_{01})\mathrm{d}A = -(y_{01} - y_0)A \quad (49)$$

$$S_{\omega 1} = \iint_A (\omega_1 - \omega_{01}) dA = - \omega_{01} A$$

(50)

$$I_{v1} = \iint_A (z - z_{01})^2 \mathrm{d}A = I_{v0} + (z_{01} - z_0)^2 A \quad (51)$$

$$I_{w1} = \iint_A (y - y_{01})^2 \mathrm{d}A = I_{w0} + (y_{01} - y_0)^2 A \quad (52)$$

$$I_{vw1} = -\iint_A (y - y_{01})(z - z_{01})\mathrm{d}A = I_{vw0} - (y_{01} - y_0)(z_{01} - z_0)A \quad (53)$$

$$I_{v\omega 1} = \iint_A (\omega_1 - \omega_{01})(z - z_{01}) dA = (z_{01} - z_0)\omega_{01} A - (a_{y1} - a_{y0}) I_{v0} - (a_{z1} - a_{z0}) I_{vw0} \quad (54)$$

$$I_{w\omega 1} = \iint_A (\omega_1 - \omega_{01})(y - y_{01}) dA = (y_{01} - y_0)\omega_{01} A + (a_{y1} - a_{y0}) I_{vw0} + (a_{z1} - a_{z0}) I_{w0} \quad (55)$$

$$I_{\omega 1} = \iint_A (\omega_1 - \omega_{01})^2 dA$$

$$= I_{\omega 0} + (a_{y1} - a_{y0})^2 I_{v0} + (a_{z1} - a_{z0})^2 I_{w0} + 2(a_{y1} - a_{y0})(a_{z1} - a_{z0}) I_{vw0} + \omega_{01}{}^2 A \quad (56)$$

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